

On the convergence of generalized power series satisfying an algebraic ODE

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Abstract

We propose a sufficient condition of the convergence of a generalized power series formally satisfying an algebraic (polynomial) ordinary differential equation. The proof is based on the majorant method.

1 Introduction

In the paper we study some properties of generalized power series

$$\varphi = \sum_{k=0}^{\infty} c_k x^{s_k}, \quad c_k \in \mathbb{C}, \quad s_0 \prec s_1 \prec \dots \in \mathbb{C}, \quad \lim_{k \rightarrow \infty} \operatorname{Re} s_k = +\infty, \quad (1)$$

formally satisfying an ordinary differential equation

$$F(x, y, \delta y, \dots, \delta^n y) = 0 \quad (2)$$

of order n with respect to the unknown y , where $F(x, y_0, y_1, \dots, y_n) \not\equiv 0$ is a polynomial of $n+2$ variables and $\delta = x \frac{d}{dx}$. Here \prec is a usual ordering by first difference: $s_k \prec s_{k+1}$ iff $\operatorname{Re} s_k < \operatorname{Re} s_{k+1}$ or $\operatorname{Re} s_k = \operatorname{Re} s_{k+1}$, $\operatorname{Im} s_k < \operatorname{Im} s_{k+1}$.

Note that substituting the series (1) into the equation (2) makes sense, as only a finite number of terms in φ contribute to any term of the form cx^s in the expansion of $F(x, \varphi, \delta\varphi, \dots, \delta^n\varphi)$ in powers of x . Indeed, $\delta^j \varphi = \sum_{k=0}^{\infty} c_k s_k^j x^{s_k}$ and an equation $s = s_{k_0} + s_{k_1} + \dots + s_{k_m}$ has a finite number of solutions $(s_{k_0}, s_{k_1}, \dots, s_{k_m})$, since $\operatorname{Re} s_k \rightarrow +\infty$. Furthermore, for any integer N an inequality $s_{k_0} + s_{k_1} + \dots + s_{k_m} \preceq N$ has also a finite number of solutions, so powers of x in the expansion of $F(x, \varphi, \delta\varphi, \dots, \delta^n\varphi)$ are well ordered with respect to \prec .

Earlier in the paper [1] generalized power series of the form (1), with $s_0 < s_1 < \dots \in \mathbb{R}$, were studied. There was proved (without the assumption $s_k \rightarrow +\infty$) that they form a differential ring, and if the series (1) satisfies the equation (2), then $\lim_{k \rightarrow \infty} s_k = +\infty$. Furthermore, the exponents $s_k \in \mathbb{R}$ of the formal solution (1) generate a finite \mathbb{Z} -module. Here we prove this fact in the case of the complex exponents $s_k \in \mathbb{C}$ (Lemma 2).

For the generalized power series (1) one may naturally define the *order*

$$\operatorname{ord} \varphi = s_0,$$

and this is also well defined for any polynomial in $\varphi, \delta\varphi, \dots, \delta^n\varphi$ with coefficients of the form αx^β , $\alpha, \beta \in \mathbb{C}$.

The main result of the paper is the following sufficient condition of the convergence of (1).

Theorem 1. *Let the generalized power series (1) formally satisfy the equation (2), $\frac{\partial F}{\partial y_n}(x, \varphi, \delta\varphi, \dots, \delta^n\varphi) \neq 0$ and*

$$\operatorname{ord} \frac{\partial F}{\partial y_j}(x, \varphi, \delta\varphi, \dots, \delta^n\varphi) \geq \operatorname{ord} \frac{\partial F}{\partial y_n}(x, \varphi, \delta\varphi, \dots, \delta^n\varphi), \quad j = 0, 1, \dots, n. \quad (3)$$

Then for any sector S of sufficiently small radius with the vertex at the origin and of the opening less than 2π , the series φ converges uniformly in S .

This theorem in a somewhat different form has been formulated in [2, Th. 3.4] for the case of the *real* exponents $s_k \in \mathbb{R}$. And in the case of the *integer* exponents it becomes Malgrange's theorem [3] on the convergence of a formal solution $\hat{\varphi} = \sum_{k=0}^{\infty} c_k x^k \in \mathbb{C}[[x]]$ of the equation (2). An idea of our proof is based on the construction of a majorant algebraic equation and was used in [4] for estimating the radius of convergence of a formal solution $\hat{\varphi} \in \mathbb{C}[[x]]$ satisfying the conditions of Malgrange's theorem. It originally comes from [5, Ch. 1, §7], similar ideas are seemed to be already appeared in [6] for studying some properties of divergent formal solutions $\hat{\varphi} \in \mathbb{C}[[x]]$ of (2).

2 Auxiliary lemmas

The first auxiliary lemma is a generalization of the corresponding lemma from [3], where it has been proved in the case of the *integer* exponents s_k . Here we use technical tools of that paper.

Lemma 1. *Under the assumptions of Theorem 1, there exists an integer $\mu' \geq 0$ such that for any integer $\mu \geq \mu'$ a transformation*

$$y = \sum_{k=0}^{\mu} c_k x^{s_k} + x^{s_\mu} u$$

reduces the equation (2) to an equation of the form

$$L(\delta)u = N(x, u, \delta u, \dots, \delta^n u), \quad (4)$$

where L is a polynomial of degree n , and N is a finite sum of monomials of the form

$$\alpha x^\beta u^{q_0} (\delta u)^{q_1} \dots (\delta^n u)^{q_n}, \quad \alpha, \beta \in \mathbb{C}, \operatorname{Re} \beta > 0, q_i \in \mathbb{Z}_+.$$

Proof. Making a transformation $y = x^{s_0} v$ and taking into consideration the equality $\delta(x^{s_0} v) = x^{s_0} (\delta + s_0) v$, we have the relations

$$\delta^i y = x^{s_0} (\delta + s_0)^i v, \quad i = 1, \dots, n,$$

and therefore come from (2) to an equation

$$H(x, v, \delta v, \dots, \delta^n v) = 0, \quad (5)$$

where the function $H(x, v_0, v_1, \dots, v_n)$ is a finite sum of monomials of the form

$$c x^{l+ms_0} v_0^{q_0} v_1^{q_1} \dots v_n^{q_n}, \quad c \in \mathbb{C}, \quad l, m, q_i \in \mathbb{Z}_+.$$

We also may assume that $l + ms_0 \geq 0$ (multiplying, if necessary, the equation (5) by a corresponding x^K , $K \in \mathbb{N}$). The obtained equation (5) has a formal solution

$$\tilde{\varphi} = \sum_{k=0}^{\infty} c_k x^{s_k - s_0}, \quad 0 \prec s_1 - s_0 \prec s_2 - s_0 \prec \dots$$

For any $\mu \in \mathbb{Z}_+$, the formal series $\tilde{\varphi}$ can be represented in the form

$$\tilde{\varphi} = \sum_{k=0}^{\mu} c_k x^{s_k - s_0} + x^{s_\mu - s_0} \psi = \varphi_\mu + x^{s_\mu - s_0} \psi, \quad \operatorname{ord} \psi = s_{\mu+1} - s_\mu \succ 0.$$

Then denoting $\Phi = (\tilde{\varphi}, \delta \tilde{\varphi}, \dots, \delta^n \tilde{\varphi}) = \Phi_\mu + x^{s_\mu - s_0} \Psi$ and applying the Taylor formula to the relation $H(x, \Phi) = 0$, we have

$$\begin{aligned} 0 &= H(x, \Phi_\mu + x^{s_\mu - s_0} \Psi) = H(x, \Phi_\mu) + x^{s_\mu - s_0} \sum_{i=0}^n \frac{\partial H}{\partial v_i}(x, \Phi_\mu) \psi_i + \\ &+ \frac{1}{2} x^{2(s_\mu - s_0)} \sum_{i,j=0}^n \frac{\partial^2 H}{\partial v_i \partial v_j}(x, \Phi_\mu) \psi_i \psi_j + \dots, \end{aligned} \quad (6)$$

where $\psi_i = (\delta + s_\mu - s_0)^i \psi$.

One can easily check that the assumption $\frac{\partial F}{\partial y_n}(x, \varphi, \delta \varphi, \dots, \delta^n \varphi) \neq 0$ of Theorem 1 implies $\frac{\partial H}{\partial v_n}(x, \Phi) \neq 0$, and the assumption (3) implies

$$\operatorname{ord} \frac{\partial H}{\partial v_i}(x, \Phi) \geq \operatorname{ord} \frac{\partial H}{\partial v_n}(x, \Phi), \quad i = 0, 1, \dots, n.$$

Let $\theta = \text{ord} \frac{\partial H}{\partial v_n}(x, \Phi)$. Then each formal series $\frac{\partial H}{\partial v_i}(x, \Phi)$ is of the form

$$\frac{\partial H}{\partial v_i}(x, \Phi) = b_i x^\theta + c_i x^{\theta_i} + \dots, \quad i = 0, 1, \dots, n,$$

where $b_i \in \mathbb{C}$ ($b_n \neq 0$), $c_i \in \mathbb{C}^*$ and $\theta_i \succ \theta$. Define a polynomial

$$L(\xi) = b_0 + b_1(\xi + s_\mu - s_0) + \dots + b_n(\xi + s_\mu - s_0)^n \quad (7)$$

of degree n and choose a number μ such that the following three conditions hold:

$$1) \text{Re}(s_\mu - s_0) > \text{Re} \theta, \quad 2) \text{Re}(s_{\mu+1} - s_\mu) > 0 \quad 3) L(\xi) \neq 0 \quad \forall \xi \succ 0.$$

Now we show that such a number μ is from the statement of the lemma.

Let us note that

$$\begin{aligned} \text{ord} \left(\frac{\partial H}{\partial v_i}(x, \Phi) - \frac{\partial H}{\partial v_i}(x, \Phi_\mu) \right) &= \text{ord} \left(x^{s_\mu - s_0} \sum_{j=0}^n \frac{\partial^2 H}{\partial v_i \partial v_j}(x, \Phi_\mu) \psi_j + \dots \right) = \\ &= \theta + \alpha_i, \quad \text{Re} \alpha_i > 0, \quad i = 0, 1, \dots, n, \end{aligned}$$

since $\text{Re}(s_\mu - s_0) > \text{Re} \theta$, and the real parts of the numbers $\text{ord} \frac{\partial^2 H}{\partial v_i \partial v_j}(x, \Phi_\mu)$ are nonnegative (the same is true for the orders of the other partial derivatives of the function H). Therefore,

$$\frac{\partial H}{\partial v_i}(x, \Phi_\mu) = b_i x^\theta + \tilde{c}_i x^{\tilde{\theta}_i} + \dots, \quad \tilde{\theta}_i \succeq \theta + \alpha_i.$$

Now from the relation (6) and conditions $\text{Re}(s_{\mu+1} - s_\mu) > 0$, $\text{Re}(s_\mu - s_0) > \text{Re} \theta$, it follows that

$$\text{Re}(\text{ord} H(x, \Phi_\mu) - (s_\mu - s_0 + \theta)) > 0.$$

Hence the relation (6) can be divided by $x^{s_\mu - s_0 + \theta}$, and we obtain the equality of the form

$$L(\delta)\psi - N(x, \psi, \delta\psi, \dots, \delta^n\psi) = 0,$$

where the polynomial L is defined by the formula (7), and N is such as in the statement of the lemma. Thus, the transformation

$$v = \sum_{k=0}^{\mu} c_k x^{s_k - s_0} + x^{s_\mu - s_0} u$$

reduces the equation (5) to the equation

$$L(\delta)u = N(x, u, \delta u, \dots, \delta^n u),$$

with a formal solution $u = \psi$. \square

Remark 1. The condition $L(\xi) \neq 0$, for all $\xi \succ 0$, is not used in the proof of Lemma 1, but we add it from the beginning as it will be used in the proof of the next lemma and in the sequel.

Let us define an additive semi-group Γ generated by a (finite) set of power exponents of the variable x containing in $N(x, u, \delta u, \dots, \delta^n u)$, and let r_1, \dots, r_ν be generators of this semi-group, that is,

$$\Gamma = \{m_1 r_1 + \dots + m_\nu r_\nu \mid m_i \in \mathbb{Z}_+, \sum_{i=1}^{\nu} m_i > 0\}, \quad \text{Re } r_i > 0.$$

The second auxiliary lemma is a consequence of the first one and describes a structure of the set of power exponents $s_k \in \mathbb{C}$ of the formal solution (1).

Lemma 2. *All the numbers $s_k - s_\mu$, $k \geq \mu + 1$, belong to an additive semi-group Γ .*

Proof. We use the fact that the generalized formal power series $\psi = \sum_{k=\mu+1}^{\infty} c_k x^{s_k - s_\mu}$ satisfies the relation

$$L(\delta)\psi = N(x, \psi, \delta\psi, \dots, \delta^n \psi). \quad (8)$$

The first term of the left hand side of (8) is $L(s_{\mu+1} - s_\mu) c_{\mu+1} x^{s_{\mu+1} - s_\mu}$ (note that $L(s_{\mu+1} - s_\mu) \neq 0$, as $s_{\mu+1} - s_\mu \succ 0$), while the first term of the right hand side of (8) is a monomial αx^β , since other monomials $\alpha x^\beta \psi^{q_0} (\delta\psi)^{q_1} \dots (\delta^n \psi)^{q_n}$, $\sum_{i=0}^n q_i > 0$, have higher orders. Therefore,

$$L(s_{\mu+1} - s_\mu) c_{\mu+1} x^{s_{\mu+1} - s_\mu} = \alpha x^\beta,$$

and $s_{\mu+1} - s_\mu = \beta \in \Gamma$.

Further, the second term of the left hand side of (8) is $L(s_{\mu+2} - s_\mu) c_{\mu+2} x^{s_{\mu+2} - s_\mu}$, while the second term of the right hand side of (8) is of the form $\alpha x^{\beta+m(s_{\mu+1}-s_\mu)}$, $m \in \mathbb{Z}_+$, $\beta \in \Gamma$. Therefore,

$$L(s_{\mu+2} - s_\mu) c_{\mu+2} x^{s_{\mu+2} - s_\mu} = \alpha x^{\beta+m(s_{\mu+1}-s_\mu)},$$

and $s_{\mu+2} - s_\mu = \beta + m(s_{\mu+1} - s_\mu) \in \Gamma$. In the analogous way, one obtains that all the numbers $s_k - s_\mu$, $k > \mu$, belong to the semi-group Γ . \square

We may assume that the generators r_1, \dots, r_ν of the semi-group Γ are linearly independent over \mathbb{Z} . This is provided by the following lemma.

Lemma 3. *There are complex numbers ρ_1, \dots, ρ_τ linearly independent over \mathbb{Z} , such that all $\operatorname{Re} \rho_i > 0$ and an additive semi-group Γ' generated by them contains the above semi-group Γ generated by r_1, \dots, r_ν .*

Proof. Let r_1, \dots, r_τ be a maximal system of linearly independent over \mathbb{Z} elements from the set $\{r_1, \dots, r_\nu\}$. It is sufficient to prove that if we add any number b with $\operatorname{Re} b > 0$ such that r_1, \dots, r_τ, b become linearly dependent over \mathbb{Z} , then the semi-group G generated by r_1, \dots, r_τ, b is contained in some semi-group Γ' generated by τ linearly independent over \mathbb{Z} complex numbers ρ_1, \dots, ρ_τ having positive real parts. We may assume that

$$b = m_1 r_1 + \dots + m_{\tau-j} r_{\tau-j} - m_{\tau-j+1} r_{\tau-j+1} - \dots - m_\tau r_\tau, \quad m_i \in \mathbb{N}, \quad 1 \leq j \leq \tau - 1, \quad (9)$$

and we prove the existence of such a semi-group Γ' by the induction with respect to the number j of the signs "−" before the coefficients in the linear combination (9). (We assume that all $m_i \neq 0$. In general, if some coefficients m_i are equal to zero, then the corresponding generators r_i are included without changes into the set of generators of a new semi-group Γ' , i. e., $\rho_i = r_i$ for such r_i . In this case the reader can easily make the corresponding changes in the reasonings below.)

For $j = 1$ we have

$$b = m_1 r_1 + \dots + m_{\tau-1} r_{\tau-1} - m_\tau r_\tau.$$

There are rational positive numbers $p_1/q_1, \dots, p_{\tau-1}/q_{\tau-1}$ such that $p_1/q_1 + \dots + p_{\tau-1}/q_{\tau-1} = 1$ and

$$\frac{p_i}{q_i} < \frac{m_i \operatorname{Re} r_i}{m_\tau \operatorname{Re} r_\tau}, \quad i = 1, \dots, \tau - 1. \quad (10)$$

Indeed, the intersection of the box

$$\{(x_1, \dots, x_{\tau-1}) \in \mathbb{R}^{\tau-1} \mid 0 < x_i < m_i \operatorname{Re} r_i / m_\tau \operatorname{Re} r_\tau\}$$

with the hyperplane

$$\pi = \{x_1 + \dots + x_{\tau-1} = 1\}$$

is a non-empty open subset of π , since $\sum_{i=1}^{\tau-1} m_i \operatorname{Re} r_i / m_\tau \operatorname{Re} r_\tau > 1$ in view of the condition $\operatorname{Re} b > 0$. Hence, we can choose a point in this intersection that has rational coordinates.

Generators ρ_1, \dots, ρ_τ of a semi-group Γ' that are linearly independent over \mathbb{Z} now can be defined as follows:

$$\begin{aligned}\rho_1 &= r_1 - \frac{p_1}{q_1 m_1} m_\tau r_\tau, \quad \dots, \quad \rho_{\tau-1} = r_{\tau-1} - \frac{p_{\tau-1}}{q_{\tau-1} m_{\tau-1}} m_\tau r_\tau, \\ \rho_\tau &= \frac{1}{(q_1 m_1) \dots (q_{\tau-1} m_{\tau-1})} r_\tau.\end{aligned}$$

Then $\operatorname{Re} \rho_i > 0$ according to (10). Furthermore $G \subset \Gamma'$, as $b = m_1 \rho_1 + \dots + m_{\tau-1} \rho_{\tau-1}$ and

$$\begin{aligned}r_1 &= \rho_1 + \frac{p_1}{q_1 m_1} m_\tau r_\tau = \rho_1 + n_1 \rho_\tau, \quad n_1 \in \mathbb{N}, \\ \dots &\quad \dots \\ r_{\tau-1} &= \rho_{\tau-1} + \frac{p_{\tau-1}}{q_{\tau-1} m_{\tau-1}} m_\tau r_\tau = \rho_{\tau-1} + n_{\tau-1} \rho_\tau, \quad n_{\tau-1} \in \mathbb{N}, \\ r_\tau &= (q_1 m_1) \dots (q_{\tau-1} m_{\tau-1}) \rho_\tau.\end{aligned}$$

For an arbitrary $j > 1$ we write the number b in the form $b = b' - m_\tau r_\tau$, where

$$b' = m_1 r_1 + \dots + m_{\tau-j} r_{\tau-j} - m_{\tau-j+1} r_{\tau-j+1} - \dots - m_{\tau-1} r_{\tau-1}$$

has $j-1$ signs "−" in its representation of the form (9). Thus, we may apply an inductive assumption to the number b' and write it as follows:

$$b' = m_1 \rho_1 + \dots + m_{\tau-j} \rho_{\tau-j},$$

where the numbers $\rho_1, \dots, \rho_{\tau-j}, \dots, \rho_{\tau-1}$ are linearly independent over \mathbb{Z} and expressed via linear combinations of $r_1, \dots, r_{\tau-1}$ with rational coefficients (conversely, $r_1, \dots, r_{\tau-1}$ are expressed via linear combinations of $\rho_1, \dots, \rho_{\tau-1}$ with positive integer coefficients). Hence, r_τ and $\rho_1, \dots, \rho_{\tau-j}, \dots, \rho_{\tau-1}$ are linearly independent over \mathbb{Z} , and we conclude for $b = m_1 \rho_1 + \dots + m_{\tau-j} \rho_{\tau-j} - m_\tau r_\tau$ as in the case $j = 1$. \square

3 Proof of Theorem 1

For the simplicity of the presentation, we give the proof in the case of two generators of the semi-group Γ :

$$\Gamma = \{lr_1 + mr_2 \mid l, m \in \mathbb{Z}_+, l + m > 0\}, \quad \operatorname{Re} r_{1,2} > 0.$$

As will be shown further, in this case we deal with functions of two variables. In the case of an arbitrary number ν of generators, as can be easily seen, the proof is analogous, only functions of higher number of variables are involved.

We should prove the convergence of the generalized formal power series

$$\psi = \sum_{k=\mu+1}^{\infty} c_k x^{s_k - s_\mu},$$

which satisfies the equality

$$L(\delta)\psi = N(x, \psi, \delta\psi, \dots, \delta^n \psi) \tag{11}$$

obtained in Lemma 1. According to Lemma 2, all the exponents $s_k - s_\mu$ belong to the semi-group Γ :

$$s_k - s_\mu = lr_1 + mr_2, \quad (l, m) \in M \subset \mathbb{Z}_+^2 \setminus \{0\},$$

for some subset M such that the map $k \mapsto (l, m)$ is a bijection from $\mathbb{N} \setminus \{1, \dots, \mu\}$ to M . Then

$$\begin{aligned}L(\delta)\psi &= \sum_{k=\mu+1}^{\infty} L(s_k - s_\mu) c_k x^{s_k - s_\mu} = \sum_{(l,m) \in M} L(lr_1 + mr_2) c_{l,m} x^{lr_1 + mr_2} = \\ &= \sum_{(l,m) \in \mathbb{Z}_+^2 \setminus \{0\}} L(lr_1 + mr_2) c_{l,m} x^{lr_1 + mr_2}\end{aligned}$$

(in the last series one puts $c_{l,m} = 0$, if $(l, m) \notin M$).

Without loss of generality we may assume that all $|s_k - s_\mu| = |lr_1 + mr_2| \geq 1$, since $\lim_{k \rightarrow \infty} (s_k - s_\mu) = \infty$. (In the opposite case we make the transformation $u = \sum_{k=\mu+1}^\nu c_k x^{s_k - s_\mu} + w$, where ν is such that all $|s_k - s_\mu| \geq 1$ for $k > \nu$. This transformation reduces the equation (4) to an equation of the same form with respect to the unknown w , with the formal solution $w = \sum_{k=\nu+1}^\infty c_k x^{s_k - s_\mu}$.)

The function $N(x, u_0, u_1, \dots, u_n)$ determining the right hand side of the equality (11) is a finite sum of the form

$$N(x, u_0, u_1, \dots, u_n) = \sum_{p,q,Q} \alpha_{p,q,Q} x^{pr_1 + qr_2} u_0^{q_0} u_1^{q_1} \dots u_n^{q_n},$$

where $\alpha_{p,q,Q} \in \mathbb{C}$, $(p, q) \in \mathbb{Z}_+^2 \setminus \{0\}$, $Q = (q_0, q_1, \dots, q_n) \in \mathbb{Z}_+^{n+1}$. Thus the equality (11) is written as follows:

$$\sum_{(l,m) \in \mathbb{Z}_+^2 \setminus \{0\}} L(lr_1 + mr_2) c_{l,m} x^{lr_1 + mr_2} = \sum_{p,q,Q} \alpha_{p,q,Q} x^{pr_1 + qr_2} \psi^{q_0} (\delta\psi)^{q_1} \dots (\delta^n\psi)^{q_n}, \quad (12)$$

where

$$\delta^j \psi = \sum_{k=\mu+1}^\infty c_k (s_k - s_\mu)^j x^{s_k - s_\mu} = \sum_{(l,m) \in \mathbb{Z}_+^2 \setminus \{0\}} (lr_1 + mr_2)^j c_{l,m} x^{lr_1 + mr_2}.$$

Therefore, we have the coincidence of the following two formal power series of two independent variables z_1, z_2 :

$$\sum_{(l,m) \in \mathbb{Z}_+^2 \setminus \{0\}} L(lr_1 + mr_2) c_{l,m} z_1^l z_2^m = \sum_{p,q,Q} \alpha_{p,q,Q} z_1^p z_2^q \psi_0^{q_0} \psi_1^{q_1} \dots \psi_n^{q_n}, \quad (13)$$

where

$$\psi_j = \sum_{(l,m) \in \mathbb{Z}_+^2 \setminus \{0\}} (lr_1 + mr_2)^j c_{l,m} z_1^l z_2^m \in \mathbb{C}[[z_1, z_2]], \quad j = 0, 1, \dots, n.$$

Indeed, the coefficient $a_{l,m}$ of a monomial $z_1^l z_2^m$ in the right hand side of (13) coincides with the coefficient of the corresponding monomial $x^{lr_1 + mr_2}$ in the right hand side of the equality (12), since for each pair $(l, m) \in \mathbb{Z}_+^2 \setminus \{0\}$ there is no another pair (l', m') such that $l'r_1 + m'r_2 = lr_1 + mr_2$ (in view of the linear independence of the numbers r_1, r_2 over \mathbb{Z}). Hence, $a_{l,m} = L(lr_1 + mr_2) c_{l,m}$.

To prove the convergence of $\psi_0 \in \mathbb{C}[[z_1, z_2]]$ in some neighbourhood of the origin, we construct an equation

$$\sigma W = \sum_{p,q,Q} |\alpha_{p,q,Q}| z_1^p z_2^q W^{q_0} W^{q_1} \dots W^{q_n}, \quad (14)$$

whose right hand side is obtained from that of the equality (13) by the change of the coefficients $\alpha_{p,q,Q}$ to their absolute values and all the ψ_j to the one variable W . The number σ is defined by the formula

$$\sigma = \inf_{k > \mu} \frac{|L(s_k - s_\mu)|}{|s_k - s_\mu|^n} = \inf_{(l,m) \in M} \frac{|L(lr_1 + mr_2)|}{|lr_1 + mr_2|^n}$$

and is a positive real number, since $L(\xi) \neq 0$ for all $\xi \succ 0$, and $\lim_{k \rightarrow \infty} |L(s_k - s_\mu)| / |s_k - s_\mu|^n = |b_n| > 0$ (recall that $L(\xi) = b_0 + b_1(\xi + s_\mu - s_0) + \dots + b_n(\xi + s_\mu - s_0)^n$, $b_n \neq 0$). The equation (14) possesses a unique holomorphic near the origin solution

$$W = \sum_{(l,m) \in \mathbb{Z}_+^2 \setminus \{0\}} A_{l,m} z_1^l z_2^m$$

satisfying the condition $W(0,0) = 0$. This follows from the theorem on implicit function. One can write the coefficients $A_{l,m}$ in the form

$$A_{l,m} = C_{l,m} |lr_1 + mr_2|^n, \quad |C_{l,m}| \leq |A_{l,m}|.$$

Further we prove that the convergent near the origin power series

$$\widetilde{W} = \sum_{(l,m) \in \mathbb{Z}_+^2 \setminus \{0\}} C_{l,m} z_1^l z_2^m$$

is majorant for the formal power series ψ_0 , that is,

$$C_{l,m} \in \mathbb{R}_+, \quad |c_{l,m}| \leq C_{l,m} \quad \forall (l,m) \in \mathbb{Z}_+^2 \setminus \{0\},$$

which will imply the convergence of ψ_0 in some neighbourhood of the origin.

First we use the equality (13) to obtain recursive expressions for the coefficients $c_{l,m}$. Denote by ϕ the formal power series from the right hand side of this equality,

$$\phi = \sum_{p,q,Q} \alpha_{p,q,Q} z_1^p z_2^q \psi_0^{q_0} \psi_1^{q_1} \dots \psi_n^{q_n} \in \mathbb{C}[[z_1, z_2]],$$

then (13) implies

$$L(lr_1 + mr_2) c_{l,m} = \frac{\partial_{z_1}^l \partial_{z_2}^m \phi}{l! m!} \Big|_{z_1=z_2=0}, \quad (15)$$

where ∂_{z_1} is the partial derivative with respect to z_1 , and ∂_{z_2} is that with respect to z_2 . To express $\partial_{z_1}^l \partial_{z_2}^m \phi(0,0)$, let us apply the formulae for the derivation of a product,

$$\begin{aligned} \partial_{z_1}^l (f_0 \dots f_n) &= \sum_{l_0 + \dots + l_n = l} \frac{l!}{l_0! l_1! \dots l_n!} (\partial_{z_1}^{l_0} f_0) \dots (\partial_{z_1}^{l_n} f_n), \quad f_i \in \mathbb{C}[[z_1, z_2]], \\ \partial_{z_1}^l \partial_{z_2}^m (f_0 \dots f_n) &= \sum_{\substack{l_0 + \dots + l_n = l \\ m_0 + \dots + m_n = m}} l! m! \frac{\partial_{z_1}^{l_0} \partial_{z_2}^{m_0} f_0}{l_0! m_0!} \dots \frac{\partial_{z_1}^{l_n} \partial_{z_2}^{m_n} f_n}{l_n! m_n!}. \end{aligned}$$

Thus, we have

$$\frac{1}{l! m!} \partial_{z_1}^l \partial_{z_2}^m (z_1^p z_2^q \psi_0^{q_0} \dots \psi_n^{q_n}) \Big|_{z_1=z_2=0} = \sum_{\substack{l_0 + \dots + l_n = l - p \\ m_0 + \dots + m_n = m - q}} \frac{\partial_{z_1}^{l_0} \partial_{z_2}^{m_0} \psi_0^{q_0}}{l_0! m_0!} \dots \frac{\partial_{z_1}^{l_n} \partial_{z_2}^{m_n} \psi_n^{q_n}}{l_n! m_n!} \Big|_{z_1=z_2=0} \quad (16)$$

for any $l \geq p$, $m \geq q$ (and $= 0$, if $l < p$ or $m < q$). Every partial derivative in this sum is expressed as follows:

$$\begin{aligned} \frac{\partial_{z_1}^{l_j} \partial_{z_2}^{m_j} \psi_j^{q_j}}{l_j! m_j!} \Big|_{z_1=z_2=0} &= \sum_{\substack{\lambda_1 + \dots + \lambda_{q_j} = l_j \\ \mu_1 + \dots + \mu_{q_j} = m_j}} \frac{\partial_{z_1}^{\lambda_1} \partial_{z_2}^{\mu_1} \psi_j}{\lambda_1! \mu_1!} \dots \frac{\partial_{z_1}^{\lambda_{q_j}} \partial_{z_2}^{\mu_{q_j}} \psi_j}{\lambda_{q_j}! \mu_{q_j}!} \Big|_{z_1=z_2=0} = \\ &= \sum_{\substack{\lambda_1 + \dots + \lambda_{q_j} = l_j \\ \mu_1 + \dots + \mu_{q_j} = m_j}} c_{\lambda_1, \mu_1} (\lambda_1 r_1 + \mu_1 r_2)^j \dots c_{\lambda_{q_j}, \mu_{q_j}} (\lambda_{q_j} r_1 + \mu_{q_j} r_2)^j \end{aligned} \quad (17)$$

(note that this expression is equal to zero, if $l_j + m_j < q_j$). Combining the formulae (16), (17) we obtain

$$\frac{\partial_{z_1}^l \partial_{z_2}^m \phi}{l! m!} \Big|_{z_1=z_2=0} = \sum_{p,q,Q} \alpha_{p,q,Q} \sum_{\substack{l_0 + \dots + l_n = l - p \\ m_0 + \dots + m_n = m - q}} \frac{\partial_{z_1}^{l_0} \partial_{z_2}^{m_0} \psi_0^{q_0}}{l_0! m_0!} \dots \frac{\partial_{z_1}^{l_n} \partial_{z_2}^{m_n} \psi_n^{q_n}}{l_n! m_n!} \Big|_{z_1=z_2=0}, \quad (18)$$

where

$$\frac{\partial_{z_1}^{l_j} \partial_{z_2}^{m_j} \psi_j^{q_j}}{l_j! m_j!} \Big|_{z_1=z_2=0} = \sum_{\substack{\lambda_1+\dots+\lambda_{q_j}=l_j \\ \mu_1+\dots+\mu_{q_j}=m_j}} (\lambda_1 r_1 + \mu_1 r_2)^j \dots (\lambda_{q_j} r_1 + \mu_{q_j} r_2)^j c_{\lambda_1, \mu_1} \dots c_{\lambda_{q_j}, \mu_{q_j}}. \quad (19)$$

The summands in the right hand side of (19) do not contain the coefficient $c_{l,m}$. Indeed, if $(\lambda_i, \mu_i) = (l, m)$ in some summand, this would necessary imply $(l_j, m_j) = (l, m)$ and, therefore, $(\lambda_s, \mu_s) = (0, 0)$ for the other (λ_s, μ_s) in this summand. Thus, the formula (15) can be written in the form

$$L(lr_1 + mr_2) c_{l,m} = p_{l,m}(\{\alpha_{p,q,Q}\}, \{c_{\lambda,\mu}\}),$$

where $p_{l,m}$ is the polynomial of the variables $\{\alpha_{p,q,Q}\}, \{c_{\lambda,\mu}\}$ (with $(\lambda, \mu) \in M, \lambda \leq l, \mu \leq m, (\lambda, \mu) \neq (l, m)$) determined by the formulae (18), (19).

Now we similarly use the equality (14) to obtain recursive expressions for the coefficients $C_{l,m}$. Denote by Φ the power series from the right hand side of this equality,

$$\Phi = \sum_{p,q,Q} |\alpha_{p,q,Q}| z_1^p z_2^q W^{q_0} W^{q_1} \dots W^{q_n} \in \mathbb{C}\{z_1, z_2\},$$

for $W = \sum_{(l,m) \in \mathbb{Z}_+^2 \setminus \{0\}} |lr_1 + mr_2|^n C_{l,m} z_1^l z_2^m$. Then (14) implies

$$\sigma |lr_1 + mr_2|^n C_{l,m} = \frac{\partial_{z_1}^l \partial_{z_2}^m \Phi}{l! m!} \Big|_{z_1=z_2=0}. \quad (20)$$

Keeping in mind the analogy and difference between the series $\phi \in \mathbb{C}[[z_1, z_2]]$ and $\Phi \in \mathbb{C}\{z_1, z_2\}$, we obtain

$$\frac{\partial_{z_1}^l \partial_{z_2}^m \Phi}{l! m!} \Big|_{z_1=z_2=0} = \sum_{p,q,Q} |\alpha_{p,q,Q}| \sum_{\substack{l_0+l_1+\dots+l_n=l-p \\ m_0+m_1+\dots+m_n=m-q}} \frac{\partial_{z_1}^{l_0} \partial_{z_2}^{m_0} W^{q_0}}{l_0! m_0!} \dots \frac{\partial_{z_1}^{l_n} \partial_{z_2}^{m_n} W^{q_n}}{l_n! m_n!} \Big|_{z_1=z_2=0}, \quad (21)$$

where

$$\frac{\partial_{z_1}^{l_j} \partial_{z_2}^{m_j} W^{q_j}}{l_j! m_j!} \Big|_{z_1=z_2=0} = \sum_{\substack{\lambda_1+\dots+\lambda_{q_j}=l_j \\ \mu_1+\dots+\mu_{q_j}=m_j}} |\lambda_1 r_1 + \mu_1 r_2|^{n_1} \dots |\lambda_{q_j} r_1 + \mu_{q_j} r_2|^{n_{q_j}} C_{\lambda_1, \mu_1} \dots C_{\lambda_{q_j}, \mu_{q_j}}. \quad (22)$$

Thus, the formula (20) can be written in the form

$$\sigma |lr_1 + mr_2|^n C_{l,m} = P_{l,m}(\{|\alpha_{p,q,Q}|\}, \{C_{\lambda,\mu}\}), \quad (23)$$

where $P_{l,m}$ is the polynomial of the variables $\{|\alpha_{p,q,Q}|\}, \{C_{\lambda,\mu}\}$ ($\lambda \leq l, \mu \leq m, (\lambda, \mu) \neq (l, m)$) with the real positive coefficients determined by the formulae (21), (22). Since for (l, m) equal to $(1, 0)$ and $(0, 1)$ we have

$$\sigma |r_1|^n C_{1,0} = \partial_{z_1} \Phi(0, 0) = |\alpha_{1,0,0}|, \quad \sigma |r_2|^n C_{0,1} = \partial_{z_2} \Phi(0, 0) = |\alpha_{0,1,0}|,$$

all the coefficients $C_{l,m}$ are real nonnegative numbers.

Finally we come to a conclusive part of the proof, the estimates

$$|c_{l,m}| \leq C_{l,m} \quad \forall (l, m) \in \mathbb{Z}_+^2 \setminus \{0\}.$$

We prove them by the induction with respect to the sum $l + m$ of the indices.

For $l + m = 1$ according to (15) we have

$$L(r_1) c_{1,0} = \partial_{z_1} \phi(0, 0) = \alpha_{1,0,0}, \quad L(r_2) c_{0,1} = \partial_{z_2} \phi(0, 0) = \alpha_{0,1,0},$$

hence¹

$$|c_{1,0}| = \frac{|\alpha_{1,0,0}|}{|L(r_1)|} = \frac{\sigma |r_1|^n C_{1,0}}{|L(r_1)|} \leq C_{1,0}, \quad |c_{0,1}| = \frac{|\alpha_{0,1,0}|}{|L(r_2)|} = \frac{\sigma |r_2|^n C_{0,1}}{|L(r_2)|} \leq C_{0,1}.$$

Further, by the construction of the polynomials $p_{l,m}$ and $P_{l,m}$, for any $(l, m) \in M$ we have

$$|p_{l,m}(\{\alpha_{p,q,Q}\}, \{c_{\lambda,\mu}\})| \leq P_{l,m}(|\alpha_{p,q,Q}|, |c_{\lambda,\mu}|)$$

(here we use the estimate $|\lambda r_1 + \mu r_2| \geq 1$ for all $(\lambda, \mu) \in M$), and the inductive assumption (the second inequality below) implies

$$\begin{aligned} |L(lr_1 + mr_2)| |c_{l,m}| &= |p_{l,m}(\{\alpha_{p,q,Q}\}, \{c_{\lambda,\mu}\})| \leq P_{l,m}(|\alpha_{p,q,Q}|, |c_{\lambda,\mu}|) \leq \\ &\leq P_{l,m}(|\alpha_{p,q,Q}|, \{C_{\lambda,\mu}\}) \leq \sigma |lr_1 + mr_2|^n C_{l,m},^2 \end{aligned} \quad (24)$$

whence the required estimates follow:

$$|c_{l,m}| \leq \frac{\sigma |lr_1 + mr_2|^n}{|L(lr_1 + mr_2)|} C_{l,m} \leq C_{l,m}$$

(for $(l, m) \in M$ we have $|L(lr_1 + mr_2)|/|lr_1 + mr_2|^n \geq \sigma$, and $c_{l,m} = 0$ for $(l, m) \notin M$).

Now it remains to note that for any sector S with the vertex at $0 \in \mathbb{C}$ and of the opening less than 2π , the terms of the series (1) are regarded as holomorphic single-valued functions in S , and to pass from the convergence of $\psi_0 = \sum_{(l,m) \in \mathbb{Z}_+^2 \setminus \{0\}} c_{l,m} z_1^l z_2^m$ to the convergence of $\varphi = \sum_{k=0}^{\infty} c_k x^{s_k}$. Let the power series ψ_0 converge in a neighbourhood of a closed polydisk $\{|z_1| \leq R, |z_2| \leq R\}$. Then there is a positive constant C such that

$$|c_{l,m}| \leq C/R^{l+m}$$

for all $(l, m) \in \mathbb{Z}_+^2 \setminus \{0\}$. If $x \in S$ is small enough for the inequalities

$$|x^{r_1}| = |x|^{\operatorname{Re} r_1} e^{-\operatorname{Im} r_1 \cdot \arg x} \leq R_1 < R, \quad |x^{r_2}| = |x|^{\operatorname{Re} r_2} e^{-\operatorname{Im} r_2 \cdot \arg x} \leq R_1 < R$$

to be held (recall that $\operatorname{Re} r_{1,2} > 0$), then

$$|c_k x^{s_k - s_\mu}| = |c_{l,m}| \cdot |x^{r_1}|^l \cdot |x^{r_2}|^m \leq C \left(\frac{R_1}{R}\right)^l \left(\frac{R_1}{R}\right)^m = C h^{l+m}, \quad l = l(k), m = m(k).$$

As $0 < h < 1$, the series $\sum_{(l,m) \in \mathbb{Z}_+^2} h^{l+m}$ converges, hence the series $\varphi = \sum_{k=0}^{\infty} c_k x^{s_k}$ converges uniformly in S for sufficiently small $|x|$.

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¹Note that if $(1, 0) \in M$, then $|L(r_1)|/|r_1|^n \geq \sigma$, and in the opposite case $c_{1,0} = 0$. The similar is true for the index $(0, 1)$.

²In the left hand side of the last inequality in (24) the indices (λ, μ) of the variables $C_{\lambda,\mu}$ belong to the set M , while in the equality (23) they belong to $\mathbb{Z}_+^2 \setminus \{0\}$. Therefore we write in (24) the inequality instead of the equality.

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